J. Fluid Mech. (1971), *vol.* 49, *part* 4, *pp.* 775–801
Printed in Great Britain

The propagation of large amplitude tsunamis across a basin of changing depth Part 1. Off-shore behaviour

By E. VARLEY AND R. VENKATARAMAN
Center for the Application of Mathematics, Lehigh University

AND E. CUMBERBATCH
Department of Mathematics, Purdue University

(Received 10 May 1971)

A theory is presented which describes the propagation of large amplitude tsunamis across a basin of variable depth in the limit when this depth is varying slowly on a scale defined by the wavelength. In part 1 only the off-shore behaviour is considered; in part 2 some features of the final run up are described.

The technique used is to regard the wave as a slowly modulated simple wave with a slowly changing Riemann invariant. One of the most significant results is that over distances where the effect of depth variation modulates the amplitude of the wave, but does not disperse it, the variations of the amplitudes of the flow variables, such as maximum surface elevation, can be calculated as functions of the undisturbed depth without knowing how this depth varies in distance and without knowing the wave profile. These variations are fully calculated.

The work continues the investigation on large amplitude acoustic pulses in stratified media described in an earlier paper by Varley & Cumberbatch (1970). It is a generalization of Whitham's work (1953) on the sonic boom.

1. Introduction

This paper describes the behaviour of a class of large amplitude shallow-water waves as they propagate over large distances into a region where the undisturbed depth is slowly varying for the waves. This work continues the investigation of large amplitude acoustic pulses in stratified media, the first part of which was described in Varley & Cumberbatch (1970). The theory developed in that paper, to describe a large amplitude pulse whose behaviour was governed by quite general systems of hyperbolic equations, was based on heuristic arguments. Here a more mathematical justification is presented for the special case of shallow-water waves. The arguments presented can readily be generalized. However, because we are dealing with a definite set of equations which are, perhaps, the simplest equations that describe finite amplitude waves in an inhomogeneous medium realistically, a detailed account of the predictions of the theory is possible which clearly illustrates its usefulness.

The specific problem considered is that of the behaviour of a tsunami as it moves over a continental shelf towards a shoreline. The values of the pertinent physical parameters are taken from Bascom (1964), and are given in §5.2. No attempt is made to justify the use of shallow-water theory to describe such waves. (The reader is referred to Carrier 1966, where this topic is discussed.) In part 1 of this paper we consider the approach of a tsunami towards a shoreline, and follow it far enough for the decrease in undisturbed depth to increase its effective amplitude to such an extent that linear theory is not applicable. In part 2, some features of the final onslaught when the tsunami crosses the shoreline will be discussed.

A plane, progressing shallow-water gravity wave, which does not contain a bore, and which is moving into an undisturbed region where the depth h^* is constant, is necessarily a simple wave (Stoker 1957). If η^* and u denote the fluid elevation and velocity, then in any such wave irrespective of the wave profile, the Riemann variable

$$S = (g(\eta^*+h^*))^{\frac{1}{2}} - \tfrac{1}{2}u = (gh^*)^{\frac{1}{2}}. \tag{1.1}$$

When h^* varies with the distance x in the direction of wave propagation, no exact integral such as (1.1) of the governing equations exists. However, as we show in §4, when $h^*(x)$ varies slowly on a scale defined by the wavelength, or equivalently, if the wave is a pulse in the sense that its duration at any station x is small compared with the Brunt–Väisälä time

$$|\omega|^{-1} = \left|\frac{d}{dx}(gh^*)^{\frac{1}{2}}\right|^{-1}, \tag{1.2}$$

the relation (1.1), with h^* varying with x, still holds to a good approximation. Typically, at the edge of the continental shelf, which is about 60 miles off-shore, $|\omega|^{-1}$ is 75 min, one half mile off-shore it is typically 7·5 min.

The structure of a pulse becomes increasingly complex as the distance it travels increases. Here, in part 1, we consider some aspects of its behaviour in three regions in all of which, to a first approximation, η^* and u are related by (1.1). In region I the effect of the variation in h^* can be neglected and the pulse behaves as a simple wave. Consequently, in this region, the pulse is not attenuated or amplified and remains sharp in the sense that conditions at the passage of a characteristic wavelet are solely determined by the signal carried by that wavelet, and are independent of the signal carried by all precursor wavelets. However, because of amplitude dispersion, the pulse may distort and generate bores. In region II, which includes region I, the relation (1.1) still holds and the pulse is still sharp. However, the variation of $h^*(x)$ cannot, in general, be neglected: it modulates the amplitude of the pulse. In region III, which includes region II, the relation (1.1) still holds, but now, in general, the signal is no longer sharp, and the characteristics are no longer the carriers of the disturbance. The three regions can be distinguished by the effect on the pulse of the variation of the Riemann variable S. In region I neither the variation of S moving with the pulse nor across it is significant. In region II the variation of S moving with the pulse, but not its variation across it, is important. Finally, in region III both the variation in S moving with the pulse and the cumulative effect of the small variations in S across it are equally important.

Motivated by the idea formulated in §1, that the waves considered can be regarded as slowly modulated simple waves, in which the Riemann variable S is slowly varying compared with the fast Riemann variable

$$F = (g(\eta^* + h^*))^{\frac{1}{2}} + \tfrac{1}{2}u, \tag{1.3}$$

in §2 we re-write the equations of shallow-water theory in a form which is optimum for the discussion of such waves. As independent variables we use x and a characteristic parameter α; as dependent variables we use $S(\alpha, x)$, $F(\alpha, x)$ and $t(\alpha, x)$. Then, in §4.1, we show that, when the variation in S across the wave (but not moving with it) can be neglected, the variation of F with h^* at a characteristic can be calculated without knowing the dependence of h^* on x. In §4.1 we show that the variation of S across the pulse is negligible, at least away from regions where the flow is critical, if the duration of the pulse $\tau(x)$ is short in the sense that

$$|\tau\omega| \ll 1. \tag{1.4}$$

The inequality (1.4) is the basic assumption used in linear geometric acoustics. Once $h^*(x)$ and the variation of F with t at some reference station $x = 0$ are known, $t(\alpha, x)$ can readily be determined by a simple quadrature. Consequently, the variation of the flow variables as functions of (t, x) in region II can readily be found.

In §5.1 we use the description of the flow in region II to justify the conjecture that in some vicinity of any station $x = x_0$, for some limited distance, the pulse behaves as a simple wave, and that the global behaviour of the pulse in region II is obtained simply by enveloping these local simple waves in an appropriate way. The various scales over which the local simple wave approximation is valid together with the signal carried by these waves, is given in terms of the signal at $x = 0$ and the variation in h^* between $x = 0$ and $x = x_0$. In §5.2 the values of these scales are given for a continental shelf with constant slope.

One of the most significant predictions of the pulse theory is that, in region II, the variations of the maximum amplitudes of the flow variables with h^* can be calculated without knowing the dependence of h^* on x, or the detailed variation of F with t at $x = 0$. These functions are only necessary when a knowledge of the distortion of the pulse profile as it propagates is of interest. In §5.3 (one of the most important sections of the paper: it can be read independently of the remainder), it is shown that the variations of these maxima, such as maximum fluid elevation and fluid speed, can be calculated from non-linear implicit algebraic relations. The results are depicted in figures 1–4. As an example, we predict that in region II, if no bores form, the maximum elevation induced at any station x by the passage of the pulse can never grow to be greater than $\frac{7}{9}$ of the value of the undisturbed depth at which it occurs. (The ratio of maximum elevation to undisturbed depth in a solitary wave is 0·78.)

In §6.1 the cumulative effect of the small change in S across the pulse is calculated in the small amplitude limit. It is shown how the usual expansion techniques of classical geometric acoustics, which are valid only for small amplitude pulses in region II, must be modified to describe conditions in region III. This is accomplished by using expansions in the characteristic parameter α and two time scales. The general terms in the expansions are found for arbitrary forms of $h^*(x)$. Since the wave amplitude and slope of the free surface remain small, the analysis of § 6.1 should be applicable to a tsunami as it crosses an ocean and begins its climb over a continental shelf.

In § 6, we consider the special case when

$$h^* \sim h_0^* \left(1-\frac{x}{L}\right)^n \quad (n > 0), \tag{1.5}$$

as the pulse approaches the shoreline at $x = L$. When $n > \frac{4}{7}$, we show that bores must always form before the shoreline is reached although when $n > 2$, so that ω is bounded at the shoreline, the effect of dispersion need not be significant. Since the effect of bores is to dissipate a wave, the question arises as to whether a small amplitude pulse which contains a weak bore can be dissipated rapidly enough for it to remain of small amplitude up to the shoreline. It is shown in § 6.2 that this is not possible. Even with bores the effective amplitude of a pulse is large as it approaches a shoreline. However, it is possible for a pulse to become completely amplitude-dispersed while it remains of small amplitude. The final climb to the shore of such fully amplitude-dispersed pulses will be discussed in part 2.

2. Shallow-water waves

The aim of this paper is to describe conditions in a class of shallow-water gravity waves as they propagate over large distances into regions where the undisturbed depth is slowly varying for the waves. If x and t denote a horizontal distance and time measure, and if $g^{-1}h(x)$ denotes the undisturbed depth, the equations governing the disturbed depth $g^{-1}a^2(t,x)$ and the fluid velocity $u(t,x)$ in these waves can be written (Stoker 1957)

$$u_{,t} + (a^2 + \tfrac{1}{2}u^2)_{,x} = h'(x), \tag{2.1}$$

and

$$a^2_{,t} + (ua^2)_{,x} = 0. \tag{2.2}$$

In (2.1) and (2.2), u, a and $h^{\frac{1}{2}}$ have the dimensions of velocity.

When $h'(x) \equiv 0$ any boreless wave moving in a direction of increasing x into a uniform region where the Riemann function,

$$S = a - \tfrac{1}{2}u = \text{constant} = s_0 \quad \text{say}, \tag{2.3}$$

is a simple wave (Stoker 1957). In such a wave $S \equiv s_0$, and the Riemann function,

$$F = a + \tfrac{1}{2}u, \tag{2.4}$$

is invariant at any one characteristic wavelet which propagates with invariant speed

$$c = a + u. \tag{2.5}$$

This implies that $F(t,x)$ can be related to its time variation, $F_0(t)$, at any reference station $x = x_0$ by the condition that

$$F(t,x) = F_0(\beta), \tag{2.6}$$

where the characteristic variable $\beta(t, x)$ is determined from the implicit relation,

$$t = \beta + \frac{2(x-x_0)}{3F_0(\beta)-s_0}. \tag{2.7}$$

We consider waves which are progressing into a region where $h(x)$ and S are both varying so slowly that conditions may be approximated over some limited distance and for some limited time by conditions in *some* simple wave. That is, waves for which the *local* effect of the small ambient stratification is simply to *perturb* the relations (2.6) and (2.7). However, the effect of the locally small ambient stratification does, in general, accumulate in both distance and time to produce a significant influence on the behaviour of the waves. For sufficiently large variations in the distance $|x-x_0|$ and the characteristic variable β, the relations (2.6) and (2.7) are not even approximately valid. *Globally* the waves behave as *slowly modulated* simple waves. Here, we restrict attention to the global behaviours of pulses. These waves travel far enough for the effects of locally small stratification to accumulate in distance, but at any station x do not last long enough for any significant accumulation in time.

3. Modulated simple waves

Motivated by the form of the simple wave relations, which are expected to be locally valid, x is taken as one independent variable and a characteristic parameter α, rather than t, as the other. As dependent variables we choose the *fast* Riemann variable $F(\alpha, x)$ and the *slow* Riemann variable $S(\alpha, x)$ in terms of which

$$a = \tfrac{1}{2}(F+S) \quad \text{and} \quad u = F-S. \tag{3.1}$$

A new dependent variable $\Omega(\alpha, X)$, the incremental arrival time, is also introduced. This is defined as

$$\Omega = \frac{dt}{d\alpha} \text{ at constant } x. \tag{3.2}$$

In terms of these variables conditions (2.1) and (2.2) can be expressed by the equations,

$$(3F-S)F_{,x} = h'(x), \tag{3.3}$$

$$\Omega_{,x} = -2(3F-S)^{-2}[3F_{,\alpha}-S_{,\alpha}], \tag{3.4}$$

and

$$S_{,\alpha} = \tfrac{1}{4}\Omega(3F-S)(F+S)^{-1}[(3S-F)S_{,x}-h'(x)]. \tag{3.5}$$

In this paper we consider boreless pulses propagating into an undisturbed region where

$$u = 0 \quad \text{and} \quad F = S = a = h^{\frac{1}{2}}. \tag{3.6}$$

Conditions in any such pulse are described in terms of the variation in F at some reference station $x = 0$ where $h = h_0$ and where the pulse lasts for time τ_0. It is convenient to tag the characteristic wavelets so that at $x = 0$ the characteristic parameter

$$\alpha = t/\tau_0, \quad \text{and hence} \quad \Omega = \tau_0. \tag{3.7}$$

In (3.7) t is measured from the arrival of the pulse at $x = 0$.

4. Non-dispersed amplitude modulated pulses

There are many equivalent ways of motivating the approximation scheme used in this paper. Some of these, and their relation to the schemes used in classical geometric acoustics and those used by Whitham (1953) in his theory of the sonic boom, were described in Varley & Cumberbatch (1970). Here a more mathematical approach is adopted. *We consider pulses which at any station x are so short that the change ΔS in the slow variable S is small compared with the local sound speed*

$$c = a+u = \tfrac{1}{2}(3F-S). \tag{4.1}$$

Although the physical interpretation of this statement is not immediately obvious, we show below that, for a pulse moving into an undisturbed region (which does not produce a critical flow), it is equivalent to the more familiar condition that the time duration of the pulse $\tau(x)$ at any station x is short in the sense that

$$|\tau\omega| \ll 1, \tag{4.2}$$

where

$$\omega(x) = dh^{\frac{1}{2}}/dx. \tag{4.3}$$

is the Brunt–Väisälä frequency of the medium. *This is the basic approximation used in the classical linear theory of geometric acoustics.* Therefore, the schemes described in this paper can be regarded as a modification of those used in linear geometric acoustics, so that they are applicable to pulses of any amplitude.

The reason that (3.3)–(3.5) are mathematically tractable when the change of S at any station x is small in the sense stated is immediately apparent when (3.3) is re-written as

$$\left[1-\frac{\Delta S}{3F-h^{\frac{1}{2}}}\right](3F-h^{\frac{1}{2}})F_{,x} = h'(x), \tag{4.4}$$

where

$$\Delta S = S-h^{\frac{1}{2}}.$$

When

$$\left|\frac{\Delta S}{3F-h^{\frac{1}{2}}}\right| \ll 1, \tag{4.5}$$

over distances where the cumulative effect of the change in S at fixed x, but not its change moving with the wave, can be neglected, (4.4) can be approximated by

$$(3F-h^{\frac{1}{2}})F_{,x} = h'(x). \tag{4.6}$$

This is a non-linear transport equation governing the variation of the fast variable F with x at each wavelet of the pulse. For any $h(x)$, (4.6) integrates to give

$$(\bar{F}-1)(\bar{F}+\tfrac{2}{3})^{\frac{2}{3}} = \phi(\alpha)H^{-\frac{5}{4}}, \tag{4.7}$$

where

$$H = h/h_0, \quad \bar{F} = Fh^{-\frac{1}{2}}, \tag{4.8}$$

and where

$$\phi(t/\tau_0) = (\bar{F}-1)(\bar{F}+\tfrac{2}{3})^{\frac{2}{3}} \quad \text{at} \quad x=0. \tag{4.9}$$

Equation (4.7) is an implicit equation for $\bar{F}$ as a function of (α, x). To determine $\bar{F}$ as a function of (t, x) it is necessary to calculate $t(\alpha, x)$ from the conditions that

$$\frac{\partial t}{\partial x} = c^{-1} = 2(3F-h^{\frac{1}{2}})^{-1}\left[1-\frac{\Delta S}{3F-h^{\frac{1}{2}}}\right]^{-1}, \quad \text{and} \quad t=\tau_0\alpha \quad \text{at} \quad x=0. \tag{4.10}$$

In the limit (4.5), over limited distances, the solution to (4.10) is approximated by

$$t = \tau_0\left[\alpha + \frac{2}{3\lambda_0}\int_0^x (\bar{F}-\tfrac{1}{3})^{-1}H^{-\frac{1}{2}}ds\right], \tag{4.11}$$

where the integration is at constant α, and where

$$\lambda_0 = \tau_0 h_0^{\frac{1}{2}} \tag{4.12}$$

is the wavelength, according to linear theory, of a wave with period τ_0.

When $\bar{F}$ can be approximated by (4.6) and (4.11), and S can be approximated by $h^{\frac{1}{2}}$, (3.4) integrates, subject to the initial condition (3.7), to give

$$\Omega = \tau_0\left[1 - \frac{4}{15\lambda_0}\phi'(\alpha)\int_0^x (\bar{F}+\tfrac{2}{3})^{-\frac{1}{2}}(\bar{F}-\tfrac{1}{3})^{-3}H^{-\frac{7}{4}}ds\right]. \tag{4.13}$$

To this approximation,

$$a = \tfrac{1}{2}h^{\frac{1}{2}}(\bar{F}+1), \quad u = h^{\frac{1}{2}}(\bar{F}-1), \quad c = \tfrac{3}{2}h^{\frac{1}{2}}(\bar{F}-\tfrac{1}{3}), \tag{4.14}$$

and the elevation $$\eta = a^2 - h = \tfrac{1}{4}h(\bar{F}+3)(\bar{F}-1). \tag{4.15}$$

The results (4.7)–(4.15), which describe conditions in a short duration boreless pulse propagating into an undisturbed region, are new. No assumption has been made about the amplitude of the disturbance. The only restriction is condition (4.5), which, according to (4.14), is equivalent to the condition that

$$|\Delta S/c| \ll 1. \tag{4.16}$$

To see that for subcritical flows it is also consistent with the more familiar condition (4.2) of geometric acoustics, note that according to (3.5) these results predict that, to a first approximation, at constant x

$$\frac{dS}{dt} = \Omega^{-1}S_{,\alpha} = -\frac{3}{8}\frac{(\bar{F}-1)(\bar{F}-\frac{1}{3})}{\bar{F}+1}h'(x), \tag{4.17}$$

so that $$\frac{\Delta S}{3F - h^{\frac{1}{2}}} = -\tfrac{1}{4}(\bar{F}-\tfrac{1}{3})^{-1}\omega(x)\int_{t_0(x)}^{t}\frac{(\bar{F}-1)(\bar{F}-\frac{1}{3})}{\bar{F}+1}ds. \tag{4.18}$$

In (4.18) the integration is at constant x;

$$t_0(x) = \int_0^x h^{-\frac{1}{2}}(s)\,ds \tag{4.19}$$

is the arrival time of the pulse at x, and $\omega(x)$ is the Brunt–Väisälä frequency. According to (4.14), for subcritical flows, when

$$c > 0, \quad \bar{F} > \tfrac{1}{3}. \tag{4.20}$$

Then, according to (4.18) and the mean value theorem,

$$\left|\frac{\Delta S}{3F - h^{\frac{1}{2}}}\right| \leqslant \tfrac{1}{4}\omega\frac{|\bar{F}_M - 1|}{\bar{F}_M + 1}(\bar{F}-\tfrac{1}{3})^{-1}\int_{t_0}^{t}(\bar{F}-\tfrac{1}{3})ds \tag{4.21}$$

$$\leqslant \tfrac{1}{4}\omega(\bar{F}-\tfrac{1}{3})^{-1}\int_{t_0}^{t}(\bar{F}-\tfrac{1}{3})ds \tag{4.22}$$

$$\leqslant \tfrac{1}{4}\omega\tau\frac{\bar{F}_M - \frac{1}{3}}{\bar{F}-\frac{1}{3}}, \tag{4.23}$$

where $$F_M(x) = \text{maximum value of} \quad F \quad \text{at} \quad x. \tag{4.24}$$

The required result now follows directly from the inequality (4.23). This implies that in regions where $(\overline{F}_M - \frac{1}{3})/(\overline{F} - \frac{1}{3})$ is bounded

$$\left|\frac{\Delta S}{3F - h^{\frac{1}{2}}}\right| = O(\omega\tau) \quad \text{as} \quad \omega\tau \to 0. \tag{4.25}$$

5. Predictions of the theory

Over distances where the cumulative effect of the change in S at fixed x can be neglected, the signal carried by the pulse is *sharp*. For, according to (4.7), (4.11), (4.14) and (4.15), just as in a simple wave conditions at any station x at the passage of the wavelet α_0 are uniquely determined by $h(x)$ and what was happening at the passage of the wavelet α_0 at some previous reference station $x = 0$. To a first approximation conditions are independent of the information carried by all the previous wavelets $0 \leqslant \alpha < \alpha_0$. In this sense, the characteristics are the carriers of the disturbance. The effect of the variation of $h(x)$ is to attenuate or amplify the amplitude of the pulse without dispersing it. In §6 it is shown that in the 'far field' the cumulative effect of the change in S at any x, as well as its change moving with the wave, must also be taken into account: it disperses the pulse.

5.1. *The local simple wave approximation*

The approximation scheme used to obtain the results described by (4.7)–(4.15) was motivated by the hypothesis that, over *some* limited distance from any station $x = x_0$, the pulse could be approximated by *some* simple wave. In any such wave F, u and a are invariant at any characteristic wavelet which propagates with invariant speed $c = a + u$. A bound on the distance over which this approximation is valid can readily be derived from the speed of the front, which is $\propto h^{\frac{1}{2}}(x)$. For, according to the mean value theorem, the change in the speed of the front at any two stations x and x_0 compared with its speed at some intermediate point z (which depends on x and x_0) can be written

$$\left|\frac{h^{\frac{1}{2}}(x) - h^{\frac{1}{2}}(x_0)}{h^{\frac{1}{2}}(z)}\right| = \frac{|x - x_0|}{l_0(z)}, \tag{5.1}$$

where $$l_0(z) = \left|\frac{2h(z)}{h'(z)}\right| \tag{5.2}$$

is the local Brunt–Väisälä length of the medium. According to (5.1), the variation in the speed of the front can only be safely neglected at *all* x in the range

$$x_0 \leqslant x \leqslant x_0 + D$$

if $$D \ll \min l_0(x) \quad \text{for} \quad x_0 \leqslant x \leqslant x_0 + D. \tag{5.3}$$

To obtain a bound on the distance over which the variation in F at a wavelet may also be neglected, note that, by the mean value theorem and (4.6),

$$\left|\frac{F(\alpha, x) - F(\alpha, x_0)}{F(\alpha, z)}\right| = \frac{|x - x_0|}{l(\alpha, z)}, \tag{5.4}$$

where $$l = \tfrac{3}{2}\bar{F}(\bar{F}-\tfrac{1}{3})l_0 \tag{5.5}$$

and z is some station between x and x_0. Linear theory would take $\bar{F} \equiv 1$ in (5.5), and hence $l \equiv l_0(z)$. Then, according to (5.4) and (5.2), the variation in F can also be neglected when (5.3) holds. Moreover, in that part of the pulse where the disturbed depth is greater than the undisturbed depth, the neglect of the variation of F over a distance D that satisfies (5.3) is also valid. This follows directly from the formula (4.15) for the elevation, which implies that, when $\eta > 0$, $\bar{F} > 1$, and hence, by (5.5), $l > l_0(z)$. Only in that part of the pulse where $\eta < 0$ and $\frac{1}{3} < \bar{F} < 1$ does the distance over which the variation in F at a wavelet can be neglected differ from the distance over which the variation in $h^{\frac{1}{2}}(x)$ can also be neglected. Consequently, when the pulse is not just a wave of elevation, condition (5.3) must be replaced by the more restrictive condition that the variations in both F and $h^{\frac{1}{2}}$ at any wavelet α may only be safely neglected at *all* x in the range $x_0 \leqslant x \leqslant x_0 + D(\alpha)$ if

$$D(\alpha) \ll \min l(\alpha, x) \quad \text{for} \quad x_0 \leqslant x \leqslant x_0 + D. \tag{5.6}$$

Since (4.14) and (4.15) express the flow variables a, u, c and η in terms of $h^{\frac{1}{2}}$ and $\bar{F} = Fh^{-\frac{1}{2}}$, their variations may also be neglected over the range (5.6).

Now that a bound has been found on the distance from $x = x_0$ over which the pulse can be approximated by a simple wave it only remains to determine the signal carried by this wave in terms of the variation in F at $x = 0$. This is easily done. For, in the approximating wave, $F(\alpha, x)$ is replaced by

$$F(\alpha, x_0) = h^{\frac{1}{2}}(x_0)\bar{F}(\alpha, x_0), \tag{5.7}$$

where $\bar{F}(\alpha, x_0)$ is given in terms of $\phi(\alpha)$, and hence in terms of $\bar{F}(\alpha, 0)$, by (4.7) with H evaluated at $x = x_0$.

To represent the approximating simple wave in the more familiar form (2.6) and (2.7), we introduce a new local characteristic parameter

$$\beta = \tau_0\left[\alpha + \frac{2}{3\lambda_0}\int_0^{x_0} (\bar{F}-\tfrac{1}{3})^{-1} H^{-\frac{1}{2}} ds\right], \tag{5.8}$$

which, according to (4.11), is the arrival time at $x = x_0$ of the wavelet α. Then, by (4.11), the arrival time of the characteristic wavelet β at $x(> x_0)$ can be written

$$t = \beta + \frac{2}{3}\int_{x_0}^{x} \frac{ds}{(F - \frac{1}{3}h^{\frac{1}{2}})}. \tag{5.9}$$

If $$F_0(\beta) \overset{\text{def}}{=} F(\alpha, x_0), \tag{5.10}$$

where $\alpha(\beta, x_0)$ is given implicitly by (5.8), then, for all x in the region (5.6), to a first approximation,

$$F = F_0(\beta) \quad \text{and} \quad t = \beta + \frac{2(x-x_0)}{3F_0(\beta) - h^{\frac{1}{2}}(x_0)}, \tag{5.11}$$

which are identical with (2.6) and (2.7) with $s_0 = h^{\frac{1}{2}}(x_0)$.

5.2. $\omega(x)$ and $l(x)$ for a continental shelf with constant slope

To obtain an idea of how long a pulse may actually last for it to be short enough for our theory to apply, consider a tsunami of elevation moving towards a shoreline over a continental shelf whose slope is constant. Then, if $x = 0$ is the edge of the continental shelf, and if $x = L$ is the shoreline,

$$H = \frac{h}{h_0} = \left(1 - \frac{x}{L}\right), \tag{5.12}$$

where $g^{-1}h_0, = h_0^*$ say, is the undisturbed depth at $x = 0$. Typical values of h_0^* and L, as given by Bascom (1964), are

$$h_0^* = 600\,\text{ft} \quad \text{and} \quad L = 60\,\text{miles} \tag{5.13}$$

With $h(x)$ given by (5.12), and the Brunt–Väisälä frequency by (4.3), the condition (4.2) on the duration of the pulse reads

$$\frac{\tau(x)}{T(x)} \ll 1, \tag{5.14}$$

where
$$T(x) = 2L(gh_0^*)^{-\frac{1}{2}}\left(1 - \frac{x}{L}\right)^{\frac{1}{2}} = [\omega(x)]^{-1} \tag{5.15}$$

is identical with the time it takes a boreless front to travel from the station x to the shoreline $x = L$. With h_0^* and L given by (5.13),

$$T(0) \doteqdot 75\,\text{min}, \tag{5.16}$$

so that, according to (5.14), a pulse is short as it passes the rim of the continental shelf if its duration is negligibly small compared with 75 min! When the pulse has reached the station $x = 0{\cdot}99L$ where $h = 0{\cdot}01h_0$, the pulse is short if $\tau \ll 0{\cdot}1T(0)$. This implies that, when h_0^* and L are given by (5.13) the pulse is still short when it is 0·6 m from the shoreline where the depth is 6 ft† if its duration is negligibly small compared with 7·5 min! Note that, according to condition (5.14), the pulse approximation is only valid at any x over a time interval which is less than the time it takes for a bore-less front to return to x after reflection from the shoreline.

With H given by (5.12), the local Brunt–Väisälä length at x (defined by (5.2)) is

$$l_0(x) = 2(L - x), \tag{5.17}$$

which is twice the distance of x from the shoreline. Since $l_0(x)$ is a decreasing function of x its minimum value in the range $x_0 \leqslant x \leqslant x_0 + D$ is $2(L - x_0 - D)$. Consequently, in this range, a wave of elevation may only be approximated by a simple wave if

$$D \ll \tfrac{2}{3}(L - x_0), \tag{5.18}$$

which is $\frac{2}{3}$ the distance from x_0 to the shoreline. This implies that, when (5.13) hold, a wave of elevation is not appreciably affected by the slope of the continental shelf ($\doteqdot \frac{1}{550}$) as it travels shoreward over the rim for distances which are small compared with 40 miles.

† This small value for the depth follows from the assumption that the continental shelf is of constant slope. The error in this assumption is most significant at the shoreline.

Of course, as the pulse moves towards the shoreline $T(x)$, and consequently $\tau(x)$, approach zero. In fact, not only does the time interval over which the pulse approximation is valid approach zero, but the assumption that the front travels with acoustic speed is also invalidated. For a theory which neglects the cumulative effect of locally small ΔS and also the cumulative effect of locally small vertical accelerations (both of which disperse the wave) predicts that a bore will always form at or near the front of a tidal wave of elevation as it moves up a beach of constant slope. The effect of a small amplitude bore and the effect of non-zero ΔS on a small amplitude pulse is discussed in §6.

5.3. *Variation of wave amplitude with h*

One of the most significant predictions of the pulse theory is that, over distances where the signal carried by the wave remains sharp, the variations in the amplitudes of the state variables with h can be calculated without knowing the forms of $h(x)$ and the signal function $\phi(\alpha)$. A knowledge of these functions is necessary only when the variations of the state variables with x at fixed t, or with t at fixed x, are of interest.

Consider, for example, the variation with h of the maximum elevation η which occurs at any station x during the passage of a wave. This maximum elevation occurs at the passage of a characteristic wavelet

$$\alpha(t,x) = \text{const.} = \alpha_M \text{ say, } \quad \text{at which} \quad \phi'(\alpha_M) = 0. \tag{5.19}$$

This is easily seen by noting that when the maximum elevation occurs

$$\eta_{,t}(t,x) = 0, \tag{5.20}$$

which, by (4.15), implies that $\overline{F}_{,\alpha}(\alpha,x) = 0$: this, together with the statement (4.7) for $\overline{F}(\alpha,x)$, implies (5.19). The variation of maximum elevation† η_M with h thus reduces to determining the variation of η with h at a characteristic. This is easily done. For (4.15) states that

$$\frac{\eta_M}{h_0} = \tfrac{1}{4}H(\overline{F}+3)(\overline{F}-1), \tag{5.21}$$

where, according to (4.7) and (4.9), the variation in $\overline{F}$ with $H = h/h_0$ is given by

$$(\overline{F}-1)(\overline{F}+\tfrac{2}{3})^{\frac{3}{2}} = (\overline{F}_0-1)(\overline{F}_0+\tfrac{2}{3})^{\frac{3}{2}}H^{-\frac{5}{4}}. \tag{5.22}$$

In (5.22) the constant parameter $\overline{F}_0$ is given in terms of the ratio η_{M_0}/h_0 of maximum elevation to undisturbed depth as the pulse passes $x = 0$, where $H = 1$, by

$$\overline{F}_0 = 2\left(1+\frac{\eta_{M_0}}{h_0}\right)^{\frac{1}{2}} - 1. \tag{5.23}$$

Once η_{M_0}/h_0 is specified, (5.21) and (5.22), with $\overline{F}_0$ determined from (5.23), determine η_M/h_0 as a function of H. The variation with H of the speed c_M of the wavelet α_M can also be determined. It is given by (4.14) as

$$c_M/h_0^{\frac{1}{2}} = \tfrac{3}{2}(\overline{F}-\tfrac{1}{3})H^{\frac{1}{2}}. \tag{5.24}$$

† The trajectory of the wavelet α_M at which $\partial(\eta(t,x))/\partial t = 0$ is not, in general, identical with that of a crest at which $\partial(\eta(t,x))/\partial x = 0$.

The description furnished by (5.21), (5.22) and (5.24), implies that, at the wavelet α_M,

$$\frac{d}{dx}\left(\frac{\eta_M}{h_0}\right) = \frac{1}{4}\frac{(\bar{F}-1)(\bar{F}-\frac{5}{3})}{\bar{F}-\frac{1}{3}}\frac{dH}{dx}, \tag{5.25}$$

and

$$\frac{d}{dx}\left(\frac{c_M}{h_0^{\frac{1}{2}}}\right) = -\frac{1}{4}\frac{\bar{F}-\frac{5}{3}}{\bar{F}-\frac{1}{3}}H^{-\frac{1}{2}}\frac{dH}{dx}. \tag{5.26}$$

According to (5.25), as a wave of elevation in which

$$\eta \geqslant 0 \quad \text{and consequently} \quad \bar{F} \geqslant 1 \tag{5.27}$$

moves into a region where the undisturbed depth is decreasing, η_M increases as it passes x, if at x

$$1 < \bar{F} < \tfrac{5}{3}, \quad \text{which corresponds to} \quad 0 < \eta/h < \tfrac{7}{9}. \tag{5.28}$$

If at $x = 0$ the maximum elevation lies in the range $0 < \eta_{M_0}/h_0 < \frac{7}{9}$, and if the pulse moves shoreward over a bottom where H is monotonically decreasing, then, according to (5.22) and (5.23), $\bar{F}$ increases monotonically and becomes unbounded as $H \to 0$. However, according to (5.21) and (5.25), even though η_M at first increases it reaches a maximum when $\bar{F} = \frac{5}{3}$, and then decreases to a finite value at the shoreline. The maximum value of η_M, η_{Mm}, is related to h_m, the value of h at which it occurs, by the simple relation

$$\eta_{Mm} = \tfrac{7}{9}h_m\dagger, \tag{5.29}$$

which is a special case of condition (5.21) with $\bar{F} = \frac{5}{3}$. h_m is determined by conditions at $x = 0$: it is given by

$$h_m/h_0 = 0{\cdot}50(\bar{F}_0-1)^{\frac{4}{5}}(\bar{F}_0+\tfrac{2}{3})^{\frac{6}{5}}, \tag{5.30}$$

where $\bar{F}_0$ is given in terms of η_{M_0}/h_0 by (5.23). Equation (5.30) is a special case of (5.22) with $\bar{F} = \frac{5}{3}$. The point at which η_M attains its maximum value only lies between $x = 0$ and the shoreline if, as the pulse passes $x = 0$, $\eta = \eta_{M_0}$ lies in the range (5.28). If $\eta_{M_0}/h_0 > \frac{7}{9}$, η_M decreases monotonically as H decreases, and reaches a limiting value η_{Ms} at the shoreline. To calculate η_{Ms} note that, as $H \to 0$, (5.22) predicts that

$$\bar{F} = (\bar{F}_0-1)^{\frac{2}{5}}(\bar{F}_0+\tfrac{2}{3})^{\frac{3}{5}}H^{-\frac{1}{2}}[1+O(\bar{F}^{-2})], \tag{5.31}$$

for all $\bar{F}_0 > 1$. This, together with (5.21), gives

$$\eta_{Ms}/h_0 = \tfrac{1}{4}(\bar{F}_0-1)^{\frac{4}{5}}(\bar{F}_0+\tfrac{2}{3})^{\frac{6}{5}} = 0{\cdot}50h_m/h_0. \tag{5.32a}$$

Figure 1 depicts typical variations in η_M/η_{M_0} with H for $\eta_{M_0}/h_0 = 0{\cdot}01$, $0{\cdot}05$ and $0{\cdot}10$. For comparison, the broken curve depicts the variation in η_M/η_{M_0} that is predicted by linear theory. This is obtained by formally linearizing (5.21) and (5.22) about $\bar{F} = 1$ to given

$$\eta_M = \eta_{M_0}H^{-\frac{1}{4}}. \tag{5.32b}$$

† This ratio, 0·78, of elevation to undisturbed depth also occurs in other contexts in the study of water waves. It is the ratio of maximum elevation to undisturbed depth of a *solitary wave* (Lamb 1945); it is also the ratio of elevation to undisturbed depth at which the wave crests of swell become unstable (Bascom 1964) and break.

The linear theory is, of course, completely erroneous at the shoreline where it predicts infinite heights. In figure 2 the variations of η_M in pulses which as they pass $x = 0$ have $\eta_{M_0}/h_0 > \frac{7}{9}$ are also illustrated. These correspond to pulses which have peaked before reaching $x = 0$.

The occurrence of a negative minimum value of η during the passage of a wave also coincides with the passage of a wavelet at which $\phi(\alpha)$ has a stationary value. Consequently, the variation of this minimum value, which we also denote by η_M, and the speed c_M of the wavelet α_M are also described by (5.21)–(5.26). Now, however, since $\eta_M < 0$, in regions where the flow remains subcritical with $c > 0$, at α_M

$$\tfrac{1}{3} < \bar{F} < 1. \tag{5.33}$$

For $\bar{F}$ in this range, according to (4.14), the fluid velocity at the passage of α_M

$$u_M < 0, \quad \text{and hence} \quad c_M < a_M. \tag{5.34}$$

Figure 3 depicts the variations in η_M/η_{M_0} when $\eta_{M_0}/h_0 = -0{\cdot}01$, $-0{\cdot}05$ and $-0{\cdot}1$. Since all these values lie in the range

$$-\tfrac{5}{9} < \eta_{M_0}/h_0 < 0 \tag{5.35}$$

(and so, by (5.23), correspond to $\bar{F}_0$ in the range $\frac{1}{3} < \bar{F}_0 < 1$) the flow induced at $x = 0$ by the passage of these waves remains subcritical with $c_M > 0$. However, as H decreases, c_M also decreases, and the flow becomes critical when $c_M = 0$ and $\bar{F} = \frac{1}{3}$ at

$$H = H_c = h_c/h_0 = 1{\cdot}38(1-\bar{F}_0)^{\frac{4}{5}}(\bar{F}_0 + \tfrac{2}{3})^{\frac{6}{5}}. \tag{5.36}$$

At this point, according to (5.21), the maximum depression

$$\eta_{Mc} = -\tfrac{5}{9}h_c. \tag{5.37}$$

Since our theory is only valid when $c > 0$, the simple relation (5.37) implies that it can only be used at any x, where the undisturbed depth is $h(x)$, for times when $\eta > -\frac{5}{9}h$. The broken curve in figure 3 gives the value of η_M/η_{M_0} at any H in the range $0 < H \leqslant 1$ at which, according to our theory, the flow becomes critical. This curve is drawn from the information provided by (5.36), (5.37) and (5.23).

To obtain the variations of η_M with H which are depicted in figures 1–3, it was implicitly assumed that the wavelet α_M, whose passage marks the occurrence of the elevation $\eta_M(x)$, reaches all x in the region of interest. Actually, any station, $x = x_c$ say, at which according to figure 3 the flow becomes critical, is not reached by the wavelet α_M whose speed $c_M \to 0$ as $x \to x_c$. What happens is that the pulse always develops a bore at some $x < x_c$. Once formed, this bore moves faster than the wavelet α_M, which it catches and overtakes. At the passage of the bore, the level of the free surface increases.

That a bore must form at some $x < x_c$ is easily seen from the expression (4.13) for the incremental arrival time Ω, which is zero at bore formation. At any time before a bore forms, there are two distinct flow regions, which are separated by the wavelet α_M at which $\phi'(\alpha_M) = 0$. In some region ahead of and bounded by α_M where $\alpha < \alpha_M$, the level of the free surface at any x is still decreasing. Since $\phi'(\alpha) < 0$, in this region Ω is strictly positive and no bores form. However, in some region behind and bounded by α_M in which $\alpha > \alpha_M$, the level of the free surface is increasing. Since $\phi'(\alpha) > 0$ in this region, and since the integral in (4.13) increases without bound as the wavelet α_M approaches x_c, Ω must have a zero, and consequently a bore must form at some x in the range $0 < x < x_c$. The point at which a bore forms depends on the behaviour of $h(x)$ and the signal function $\phi(\alpha)$. This point can be arbitrarily close to $x = x_c$ if $\phi'(\alpha)$ is sufficiently small.

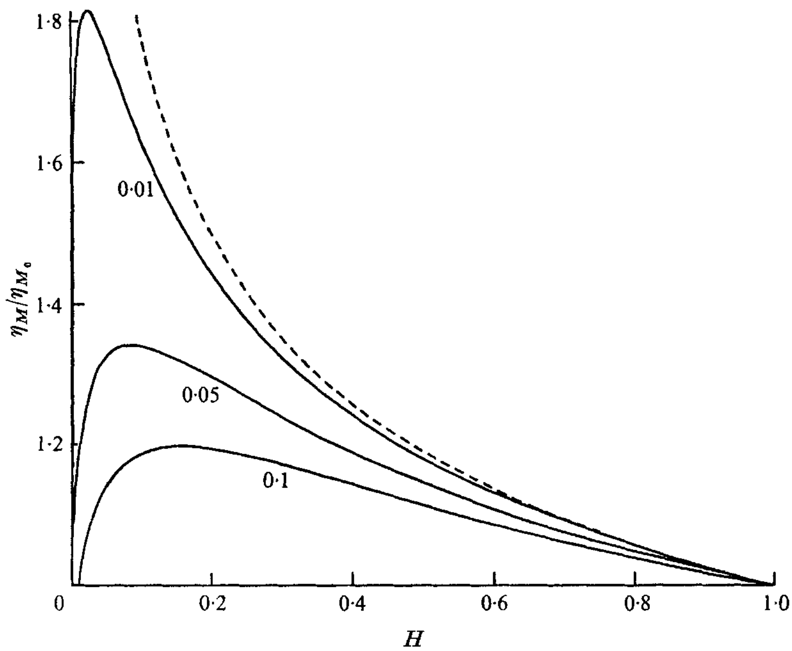

FIGURE 1. Variations of η_M/η_{M_0}, the maximum amplitude of η/η_0 in a pulse, with $H = h/h_0$ for the cases when $\eta_{M_0}/h_0 = 0{\cdot}01$, $0{\cdot}05$ and $0{\cdot}1$. The broken curve is the prediction of linear theory.

Once formed, a bore acts as a moving boundary which reflects part of the energy of the incident α-wave. Then the progressing wave approximation is valid only at any point up until a time when this reflected energy has a negligible effect on the flow. However, even for these times, the ambient slow variables $S_0(x)$ cannot, in general, be approximated by $h^{\frac{1}{2}}(x)$, as it was for boreless pulses. It must be determined in terms of conditions ahead of the bore and the jump conditions across it. This is discussed in part 2 of this paper.

According to (5.23) and (5.36), as the maximum depression at $x = 0$, η_{M_0}, approaches zero the sonic, or critical, point x_c approaches the shoreline $x = L$, where $h(L) = 0$. However, to analyse the behaviour of a pulse as it passes $x = L$ and climbs an initially dry beach, not only the complications associated with the shoreline being a sonic point must be taken into account, but also other complications associated with the fact that $h(x) \to 0$ as $x \to L$. In general, both the cumulative effect of dispersion and the reflected wave influence the pulse during run up.

In the same way that the *algebraic* relations (5.21)–(5.23) determine the variation of the maximum displacement of the free surface as a function of H, and

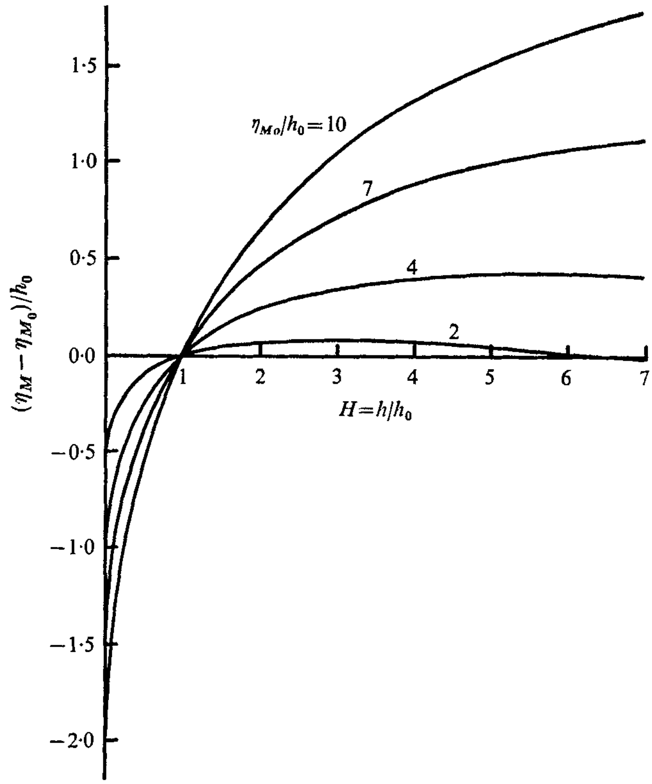

FIGURE 2. Variations of $(\eta_M - \eta_{M_0})/h_0$, the maximum value of $(\eta - \eta_0)/h_0$ in a pulse, with H when $\eta_{M_0}/h_0 = 2, 4, 7, 10$.

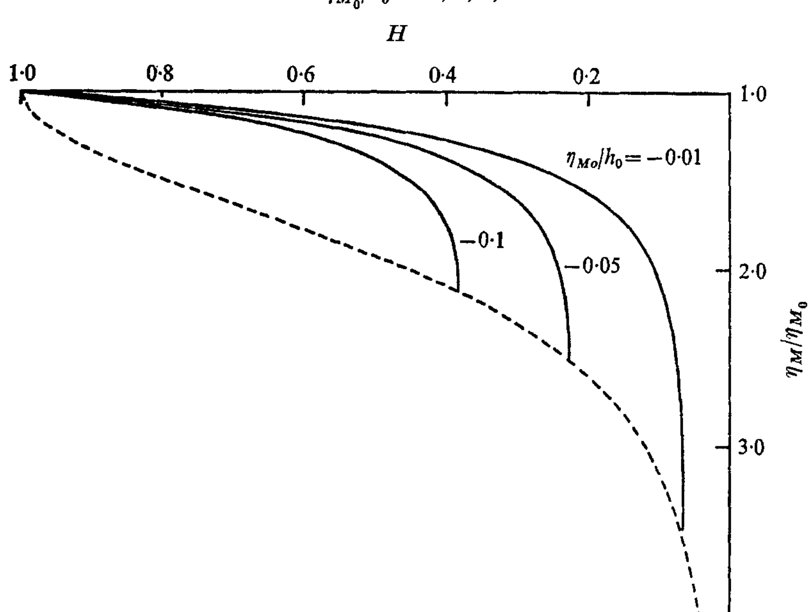

FIGURE 3. Variations in η_M/η_{M_0}, the maximum depression of the free surface, with H when $\eta_{M_0}/h_0 = -0{\cdot}01$, $-0{\cdot}05$ and $-0{\cdot}1$. The broken curve depicts the value of η_M/η_{M_0} at any H at which the flow becomes critical.

consequently as a function of x once $H(x)$ is specified, the algebraic relations (4.14), (5.22) and (5.23) also determine the variations in the maxima of the fluid velocity u_M, and rate of mass flow which is proportional to $Q_M = u_M a_M^2$. These maxima also occur at the passage of the wavelet α_M. Figure 4 depicts typical variations of the speed c_M with which these maxima travel. Since (4.14) and (5.22) predict that

$$\frac{d}{dx}\left(\frac{u_M}{h_0^{\frac{1}{2}}}\right) = -\frac{1}{2}\left(\frac{\bar{F}-1}{\bar{F}-\frac{1}{3}}\right) H^{-\frac{1}{2}}\frac{dH}{dx}, \tag{5.38}$$

and

$$\frac{d}{dx}\left(\frac{Q_M}{h_0^{\frac{3}{2}}}\right) = \tfrac{1}{8}(\bar{F}^2-1)\,H^{\frac{1}{2}}\frac{dH}{dx}, \tag{5.39}$$

as a pulse of elevation moves shoreward into a region where H is decreasing, u_M increases and Q_M decreases. At the shoreline,

$$\frac{u_{Ms}}{h_0^{\frac{1}{2}}} = 2\left(\frac{\eta_{Ms}}{h_0}\right)^{\frac{1}{2}} \quad\text{and}\quad \frac{Q_{Ms}}{h_0^{\frac{3}{2}}} = 2\left(\frac{\eta_{Ms}}{h_0}\right)^{\frac{3}{2}}, \tag{5.40}$$

where η_{Ms}/h_0 is given in terms of η_{M0}/h_0 by (5.23) and (5.32).

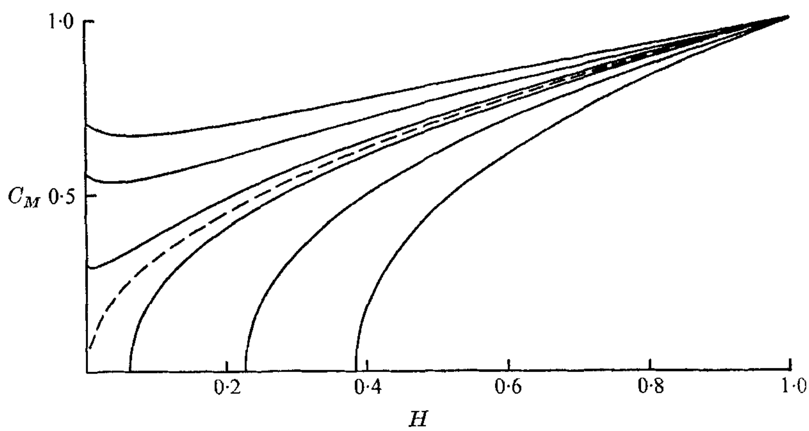

FIGURE 4. Variations in the speed C_M with H with which maximum elevations and maximum depressions travel when $\eta_{M_0}/h_0 = \pm 0{\cdot}01, \pm 0{\cdot}05$, and $\pm 0{\cdot}1$. The broken curve gives the prediction of linear theory.

6. Small amplitude theory

For many purposes the description given in §5 of how the maxima of the flow quantities vary with x is sufficient, if it is supplemented by the formula

$$\Delta t_m = \int_0^x \frac{ds}{c_M(s)}, \tag{6.1}$$

which gives the time these maxima arrive at x after passing $x = 0$. However, if a more detailed account of the behaviour of the pulse is required, such as the change in shape of the wave profile, a more thorough study of the formulae (4.7)–(4.9), (4.14), (4.15), and the formula (4.11), for $t(\alpha, x)$, must be made. This study is greatly simplified when the amplitude of the pulse is small in the sense that

$$|\bar{F}-1| \ll 1. \tag{6.2}$$

In the small amplitude limit, to a first approximation the inequality (4.21) reads

$$\left|\frac{\Delta S}{3F-h^{\frac{1}{2}}}\right| \leqslant \tfrac{1}{8}|\omega\tau|\mu, \tag{6.3}$$

where

$$\mu = \max|\bar{F}_M - 1| \ll 1. \tag{6.4}$$

Consequently, the error in approximating c by $(3F-h^{\frac{1}{2}})$ in the transport equation (3.3) for F, and in (4.10) for t, is $O(\mu|\omega\tau|)$, rather than $O(|\omega\tau|)$ as for a finite amplitude wave. This implies that over distances where the pulse remains sharp, the errors in F and t, that are obtained by neglecting the effect of non-zero ΔS, are $O(\mu|\omega\tau|)$. Of course, since the amplitude of $(\bar{F}-1)$ is itself $O(\mu)$, the error in using formulae (4.14) and (4.15) for the flow variables η and u is still $O(|\omega\tau|)$.

In the small amplitude limit, it is convenient to work with

$$\bar{f} = \bar{F} - 1, \tag{6.5}$$

rather than $\bar{F}$. Then, to a first approximation, (4.7) reads

$$\bar{f} = f_0(\alpha)H^{-\frac{5}{4}}, \tag{6.6}$$

while (4.13)–(4.15) imply that

$$\frac{\eta}{h_0} = f_0(\alpha)H^{-\frac{1}{4}}, \quad \frac{u}{h_0^{\frac{1}{2}}} = f_0(\alpha)H^{-\frac{3}{4}}, \tag{6.7}$$

and

$$\Omega = \tau_0\left[1-\frac{3}{2}\frac{f_0'(\alpha)}{\lambda_0}\int_0^x [H(s)]^{-\frac{7}{4}}ds\right]. \tag{6.8}$$

Equation (4.11) for the arrival time is approximated by

$$t - t_0(x) = \tau_0\left[\alpha-\frac{3}{2}\frac{f_0(\alpha)}{\lambda_0}\int_0^x [H(s)]^{-\frac{7}{4}}ds\right], \tag{6.9}$$

where $t_0(x)$, the arrival time of the pulse front at x, is given by (4.19). Once $f_0(\alpha)$ is known, (6.7)–(6.9) give a complete parametric description of conditions in any small amplitude pulse. To determine f_0 note that, according to (6.7) and (6.9),

$$\text{at} \quad x = 0, \quad \frac{\eta}{h_0} = f_0\left(\frac{t}{\tau_0}\right). \tag{6.10}$$

The small amplitude theory described by (6.6)–(6.9) is non-linear, since it takes into account the cumulative effect of locally small non-linearity. The small amplitude linear theory neglects the integral terms in (6.8) and (6.9). This is permissible in the *small acceleration*, or *small slope*, limit when

$$\frac{3}{2}\left|\frac{f_0'(\alpha)}{\lambda_0}\right|\left|\int_0^x [H(s)]^{-\frac{7}{4}}ds\right| \ll 1. \tag{6.11}$$

The corrections to this linear theory furnished by (6.6)–(6.9) were first suggested by Whitham (1957) in his theory of the sonic boom. *Regular* asymptotic expansions, which yield the expressions (6.6)–(6.9) as the first terms, have been given by Varley & Cumberbatch (1966) and Seymour & Varley (1970). Note that, according to (6.7) and (6.11), the fluid acceleration induced at the passage of the front is

$$u_{,t} = u_{,t}(0)\,H^{-\frac{3}{4}}\left[1-\frac{3}{2}\frac{u_{,t}(0)}{h_0}\int_0^x [H(s)]^{-\frac{7}{4}}ds\right], \tag{6.12}$$

where
$$u_{,t}(0) = \frac{h_0}{\lambda_0} f_0'(0). \tag{6.13}$$

The result (6.12) is identical with the exact expression for the acceleration given by Varley & Cumberbatch (1965).

6.1. *The cumulative effect of frequency dispersion*

In the small amplitude, small acceleration limit, when (6.2) and (6.11) hold, the cumulative effect of non-zero ΔS can readily be calculated. For then the exact equations (3.3)–(3.5) can be formally linearized about $F = S = h^{\frac{1}{2}}$ and $\Omega = \tau_0$. If

$$f = (Fh^{-\frac{1}{2}} - 1)\left(\frac{h}{h_0}\right)^{\frac{5}{4}} \quad \text{and} \quad s = (Sh^{-\frac{1}{2}} - 1)\left(\frac{h}{h_0}\right)^{\frac{5}{4}} \tag{6.14}$$

are used as dependent variables, while the linear characteristic variables

$$\alpha = t - t_0(x)/\tau_0 \tag{6.15}$$

and t_0 (the arrival time of the pulse at x) are used as independent variables, these linear equations can be written

$$f_{,t_0} = \tfrac{1}{2}\omega(t_0)s \quad \text{and} \quad s_{,\alpha} + \tfrac{1}{4}\tau_0\omega(t_0)f = \tfrac{1}{2}\tau_0 s_{,t_0}. \tag{6.16}$$

In (6.16) the Brunt–Väisälä frequency ω is considered as a function of t_0. The characteristic parameter α varies in the range

$$0 \leqslant \alpha \leqslant 1. \tag{6.17}$$

Over distances where the pulse remains sharp, equations (6.16) have asymptotic solutions (see Seymour & Varley 1970) of the form

$$f = \sum_0^\infty \tau_0^n f_n(\alpha)\tilde{P}_n(t_0), \tag{6.18}$$

and
$$s = \tau_0 \sum_0^\infty \tau_0^n f_{n+1}(\alpha)\tilde{Q}_n(t_0),$$

so that
$$s = O(\tau_0)f \quad \text{as} \quad \tau_0 \to 0. \tag{6.19}$$

In (6.18), $f_0(\alpha)$ is given in terms of the variation in η at $x = 0$ by (6.10) and

$$f_n(\alpha) = \frac{1}{(n-1)!}\int_0^\alpha (\alpha - s)^{n-1} f_0(s)\,ds \quad (n \geqslant 1). \tag{6.20}$$

The first terms in the asymptotic expansions (6.18) for f and s are

$$f_0(\alpha) \quad \text{and} \quad -\tfrac{1}{4}(\omega\tau_0)\int_0^\alpha f_0(s)\,ds, \tag{6.21}$$

respectively, the second term in the expansion for f is

$$-\tfrac{1}{8}\tau_0\left(\int_0^{t_0}\omega^2 dr\right)\int_0^\alpha f_0(s)\,ds. \tag{6.22}$$

In the small acceleration limit (6.11), the approximation to the flow variables given in (6.6)–(6.9) amounts to taking $f = f_0(\alpha)$ and $s = 0$. According to (6.21) and (6.22), for this approximation to be valid, not only is it *necessary* that the pulse be short in the sense that

$$|\tau_0\omega| \ll 1, \tag{6.23}$$

but it is also *necessary* that

$$y = \tfrac{1}{8}\tau_0 \int_0^{t_0} \omega^2 dr \ll 1. \tag{6.24}$$

Condition (6.23) is a *local* condition on the duration of the pulse, (6.24) is a *global* condition on the cumulative effect on f of small, but non-zero, s. Since y is an increasing function of t_0, as a pulse moves over large distances into a region where $\omega \neq 0$ and (6.23) holds, condition (6.24) can be violated.

When condition (6.24) does not hold but the pulse is still short, so that (6.23) does, the pulse is no longer sharp in the sense described in §5: it is frequency dispersed. However, the effect of reflected energy is still negligible in the sense that $s = O(|\omega\tau_0|)$. In this limit, f and s do not admit of regular asymptotic expansions when considered as functions of t_0 and α. However, they do have regular asymptotic expansions when considered as functions of t_0, α and the *slow variable* y. In fact these expansions are of the form

$$f = \phi_0(\alpha, y) + \sum_{n=1}^{\infty} \tau_0^n P_n(t_0)\,\phi_n(\alpha, y), \tag{6.25}$$

and

$$s = \sum_{n=1}^{\infty} \tau_0^n Q_n(t_0)\,\phi_n(\alpha, y), \tag{6.26}$$

where ϕ_0 satisfies the telegraph equation

$$\phi_{0,\alpha y} + \phi_0 = 0, \tag{6.27}$$

and

$$\phi_n = \frac{1}{(n-1)!}\int_0^{\alpha} (\alpha - s)^{n-1}\phi_0(s, y)\,ds \tag{6.28}$$

(which also satisfies the telegraph equation). The P_n and Q_n satisfy the recurrence relations,

$$\frac{dP_n}{dt_0} = \tfrac{1}{8}\omega^2 P_{n-1} + \tfrac{1}{2}\omega Q_n \quad (n \geqslant 1), \tag{6.29}$$

and

$$Q_n = \frac{1}{2}\frac{dQ_{n-1}}{dt_0} - \tfrac{1}{4}\omega P_{n-1} - \tfrac{1}{8}\omega^2 Q_{n-2} \quad (n \geqslant 1). \tag{6.30}$$

Equations (6.29) and (6.30) are solved subject to the conditions,

$$P_0 \equiv 1, \quad Q_{-1} = Q_0 \equiv 0, \tag{6.31}$$

and

$$P_n(0) = 0 \quad \text{all} \quad n \geqslant 1. \tag{6.32}$$

According to conditions (6.29)–(6.32),

$$P_1 \equiv 0, \quad P_2 = \tfrac{1}{32}[\omega_0^2 - \omega^2], \quad Q_1 = -\tfrac{1}{4}\omega \quad \text{and} \quad Q_2 = -\tfrac{1}{8}d\omega/dt_0, \tag{6.33}$$

where $\omega_0 = \omega(0)$. To confirm that the expansions (6.25) and (6.26) do indeed (formally) satisfy (6.16), use (6.27) and (6.28), which are solved subject to the conditions that

$$\phi_0 = f_0(\alpha) \quad \text{when} \quad y = 0, \quad \text{and} \quad \phi_0 \equiv 0 \quad \text{when} \quad \alpha = 0, \tag{6.34}$$

$$\phi_{n,y} = -\phi_{n+1}, \quad \phi_{n,\alpha} = \phi_{n-1}. \tag{6.35}$$

A more detailed study of the expansions (6.25) and (6.26) indicates that they provide a good representation of f and s when (6.23) holds, when $y|\tau_0\omega|$ is finite, and when the amplitudes of all derivatives of $\tau_0\omega$ with respect to t_0/τ_0 are also negligibly small compared with unity.

The solution to (6.27), that satisfies conditions (6.33), is

$$\phi_0 = \int_0^\alpha J_0(2[(\alpha-s)y]^{\frac{1}{2}})f_0'(s)\,ds, \tag{6.36}$$

where J_0 denotes the Bessel function of zero order. More generally,

$$\phi_n = \int_0^\alpha J_0(2[(\alpha-s)y]^{\frac{1}{2}})f_n'(s)\,ds, \tag{6.37}$$

where f_n is given in terms of f_0 by (6.20). According to (6.36), when

$$\alpha y = \tfrac{1}{8}(t-t_0)\int_0^{t_0}\omega^2\,dr \tag{6.38}$$

is not negligibly small compared with unity, the pulse is dispersed in the sense that, to a first approximation, conditions at any station x at the passage of the wavelet α now also depend on the information carried by all precursor wavelets. Only for a time interval $(t-t_0)$, which becomes vanishingly small as $y\to\infty$, is $\alpha y \ll 1$, which is a sufficient condition for $\phi_n(\alpha, y)$ to be approximated by $f_n(\alpha)$ (its value in the sharp pulse expansion (6.18)). Since $0 \leqslant J_0 \leqslant 1$, for all y,

$$|\phi_n(\alpha, y)| \leqslant \int_0^\alpha |f_n'(s)|\,ds. \tag{6.39}$$

If $|f_n'(\alpha)|$ is bounded, the ultimate effect of dispersion as $y\to\infty$ is to attenuate $\phi_n(\alpha, y)$. For example, if the pulse front is an acceleration front then, according to (6.36), for $0 \leqslant \alpha \leqslant 1$,

$$\text{as}\quad y\to\infty,\quad \phi_0 \sim (\alpha y)^{-\frac{1}{2}}J_1(2(\alpha y)^{\frac{1}{2}})f_0'(0)\alpha, \tag{6.40}$$

where J_1 denotes the Bessel function of order one.† In a layer near the front where $(\alpha y)\to 0$ as $y\to\infty$, the right-hand side of equation (6.40) has the limiting value $f_0'(0)\alpha$. Behind the front where $(\alpha y)\to\infty$ as $y\to\infty$, the dominant approximation to the right-hand side of (6.40) is

$$\pi^{-\frac{1}{2}}(\alpha y)^{-\frac{3}{4}}\cos[2(\alpha y)^{\frac{1}{2}} - \tfrac{3}{4}\pi]f_0'(0)\alpha. \tag{6.41}$$

Note that, according to (6.40), the asymptotic decay in ϕ_0 is controlled by $f_0'(0)$ and is independent of the detailed behaviour of $f_0(\alpha)$. Also note that, since ϕ_n is related to f_n in the same way that ϕ_0 is related to f_0, and since by the construction (6.20)

$$f_n'(0) = 0 \quad \text{for all} \quad n \geqslant 1, \tag{6.42}$$

$$\text{as}\quad y\to\infty,\quad \phi_n = o(\phi_0) \quad \text{for all} \quad n\geqslant 1, \tag{6.43}$$

so that the first term in the expansion (6.25) remains dominant.

† Dr P. A. Blythe has pointed out that this is the first term in the asymptotic expansion,

$$\phi_0 = \sum_{n=1}^{\infty}\left(\frac{\alpha}{y}\right)^{\frac{1}{2}n} J_n(2(\alpha y)^{\frac{1}{2}})f_0^n(0),$$

which is valid when $\alpha < 1$ as $(\alpha/y)\to 0$.

For non-zero αy, the first approximation to η and u in a small amplitude, small acceleration pulse are given by (6.7), with $f_0(\alpha)$ replaced by $\phi_0(\alpha, y)$. One of the consequences of this is to modify the arrival time and value of stationary values of the elevation η. Such a stationary value now occurs at any station x when $\phi_{0,\alpha} = 0$. When $y \ll 1$, the pulse is sharp, and the stationary values occur at the passage of the wavelet α_M at which $f_0'(\alpha_M) = 0$. By contrast, as $y \to \infty$, the pulse is completely dispersed, and only the value of $f_0'(0)$ influences the dominent behaviour of the pulse, Then, since ϕ_0 can be approximated by

$$\phi_0 = f_0'(0)\, y^{-1}(\alpha y)^{\frac{1}{2}} J_1(2(\alpha y)^{\frac{1}{2}}), \tag{6.44}$$

the arrival time of a local maximum, or minimum, of η is given by

$$\alpha y = \tfrac{1}{8}(t-t_0)\int_0^{t_0} \omega^2 dr = \text{constant}, = \lambda \quad \text{say}, \tag{6.45}$$

where

$$J_0(2\lambda^{\frac{1}{2}}) = 0. \tag{6.46}$$

According to (6.44)–(6.46), as $y \to \infty$, the arrival time of any one of these maxima approaches the arrival time $t_0(x)$ of the pulse front, while ϕ_{0M} decays like y^{-1}. As $y \to \infty$, the first approximations to η and u are

$$\frac{\eta}{h} = \frac{u}{h^{\frac{1}{2}}} = f_0'(0)\left(\frac{h}{h_0}\right)^{-\frac{5}{4}} y^{-1}[(\alpha y)^{\frac{1}{2}} J_1(2(\alpha y)^{\frac{1}{2}})] \tag{6.47}$$

so that, moving according to the law (6.45),

$$\frac{\eta_M}{h} \propto \left(\frac{h}{h_0}\right)^{-\frac{5}{4}} y^{-1}. \tag{6.48}$$

As y increases and the wave is dispersed, the time interval at any station x, over which there is an appreciable disturbance, also increases, until the wave can no longer be regarded as a short duration pulse. Then the pulse approximation, which is described by the expansions (6.25) and (6.26), is valid only at the head of the wave, where

$$\alpha|\tau_0\omega|^2 = o(|\tau_0\omega|) \quad \text{as} \quad |\tau_0\omega| \to 0. \tag{6.49}$$

In this region, according to (6.47), as $y \to \infty$ the flow approaches the asymptotic state (6.47), which is independent of the detailed form of the signal function $f_0(\alpha)$. At any station x, the head wave finally adopts an undular form, with an amplitude which increases with $(t-t_0)$. According to (6.41), which is the limiting behaviour of (6.47) as $(\alpha y) \to \infty$, the amplitude of the oscillations in this head wave increases like $(t-t_0)^{\frac{1}{4}}$. This increase is, of course, finally arrested by dispersion. Since the argument of the function in square brackets in (6.47) is $\alpha y = (t-t_0)\tau_0^{-1}y$, and since the coefficient of this function is independent of t, the result (6.47) states that the profile, at any station x_0 of the head of a small amplitude wave which has an acceleration front and has travelled sufficiently far, adopts the universal form $A(t-t_0)J_1(B(t-t_0)^{\frac{1}{2}})$ for some constants A and B. In part 2 we consider the run-up of a pulse with this particular profile towards a shoreline.

One of the necessary conditions that the linear analysis above give an approximate description of conditions in a pulse, or the head of a wave, is that

$$|\eta/h|(= |u/h^{\frac{1}{2}}|) \ll 1. \tag{6.50}$$

Consequently, since the maximum amplitude of $|\eta/h|$ is, by (6.41) and (6.47), proportional to $y^{-\frac{3}{4}}(h/h_0)^{-\frac{5}{4}}$ as $y \to \infty$, a necessary condition that the linear analysis is valid is that

$$y^{-1}(h/h_0)^{-\frac{5}{3}} \quad \text{is bounded as} \quad y \to \infty. \tag{6.51}$$

The behaviour of a pulse as it approaches a shoreline $x = L$ is determined, to a large measure, by whether the Brunt–Väisälä frequency $\omega(x)$ is finite or infinite as $x \to L$. When $\omega(x)$ remains bounded, so that at the shoreline the bottom topography is gently sloping in the sense that

$$dh/dx = O(h^{\frac{1}{2}}) \quad \text{as} \quad x \to L, \tag{6.52}$$

there need be no significant reflected wave, and there need be no significant cumulative effect of dispersion. However, because y remains bounded condition (6.51) is violated and consequently non-linearity plays a dominant role. In fact, a bore must always form somewhere in the pulse. When ω is unbounded at the *shoreline*, *the local effect of reflected energy is always important*, and, since y is also unbounded as $x \to L$, so is the effect of dispersion. Non-linearity may, or may not, be important. The complex problem associated with locally large ω will be discussed in a future paper.

To illustrate the various possibilities consider the case when

$$H = \frac{h}{h_0} = \left(1 - \frac{x}{L}\right)^{2m}, \quad \text{where} \quad m > 0. \tag{6.53}$$

Then,

$$\omega = -m\frac{h_0^{\frac{1}{2}}}{L}H^{(m-1)/2m}, \quad t_0 = \frac{L}{(m-1)h_0^{\frac{1}{2}}}[H^{(1-m)/2m} - 1] \tag{6.54}$$

and

$$y = \frac{1}{8}\frac{m^2}{(m-1)}\frac{\lambda_0}{L}[1 - H^{(m-1)/2m}].$$

According to (6.54), when $m > 1$, both ω and y remain finite as the shoreline is approached, although the front speed approaches zero so quickly that the front cannot reach the shoreline in a finite time. Consequently, t_0 is unbounded as the shoreline is approached. However, since for $m > 1$ the *amplitude dispersion length*

$$d = \int_0^x [H(s)]^{-\frac{7}{4}}\, ds = \frac{2L}{2 - 7m}[1 - H^{(2-7m)/4m}] \tag{6.55}$$

in (6.9) is unbounded as $H \to 0$, bores must always form in the pulse before it reaches the shoreline. The effect of weak bores is discussed in section (6.2). Their behaviour as they approach a shoreline when H is given by (6.53) will be described in part 2. When $m < 1$, which includes the case $m = \frac{1}{2}$ of a constant sloping beach, both ω and y are unbounded at the shoreline, so that both the effect of reflected energy and dispersion are locally significant. Since, however, both the condition (6.51) and the small acceleration condition (6.11) need not be violated for abruptly sloping topographies with $m < \frac{2}{7}$, it would appear that, for such shorelines, a linear theory which takes into account both reflected energy and dispersion could be uniformly valid for low-amplitude incoming pulses.

6.2. *Weak bores*

To illustrate the effect of weak bores on a pulse, we consider the *special* case of a small amplitude pulse which forms a bore before the effect of frequency dispersion is significant.

In the small amplitude limit, when amplitude dispersion is important but frequency dispersion is not, the flow is described by (6.7)–(6.9). In this limit, both the restrictions (6.50) and (6.24) hold but the small slope restriction (6.11) need not. When H is given by (6.53), a necessary condition that this be so is that

$$\delta = 2m\lambda_0/L \ll 1. \tag{6.56}$$

At bore formation, the incremental arrival time

$$\Omega = \tau_0\left[1-\frac{3}{2}\frac{f_0'(\alpha)}{\lambda_0}\int_0^x H^{-\frac{7}{4}}\,ds\right] = 0. \tag{6.57}$$

Bores form at wavelets where $f_0'(\alpha)$ has a local positive maximum. Once formed, a bore overtakes the slower-moving characteristic wavelets ahead of it, and is itself caught by the faster-moving characteristic wavelets behind it, until the wave profile between neighbouring bores is completely amplitude dispersed. If $t = B(x)$ denotes the trajectory of a bore, then the usual bore conditions (see Stoker 1957) imply that

$$h^{\frac{1}{2}}B'(x) = \frac{[1+\theta]}{[v(1+\theta)]} = \frac{[v(1+\theta)]}{[(1+\theta)\{v^2+\frac{1}{2}(1+\theta)\}]}, \tag{6.58}$$

where $$v = u/h^{\frac{1}{2}} \quad \text{and} \quad \theta = \eta/h, \tag{6.59}$$

where $[f]$ denotes the jump in any variable f at the passage of the bore. In terms of v and θ, the jump in the slowly varying Riemann function S is

$$[S] = h^{\frac{1}{2}}[(1+\theta)^{\frac{1}{2}}-\tfrac{1}{2}v]. \tag{6.60}$$

In particular, conditions (6.58) imply that, for a bore moving into an undisturbed region where $v = \theta = 0$, v behind the bore is given in terms of θ by

$$v = \theta\left(\frac{1+\frac{1}{2}\theta}{1+\theta}\right)^{\frac{1}{2}}, \tag{6.61}$$

while $$h^{\frac{1}{2}}B'(x) = (1+\theta)^{-\frac{1}{2}}(1+\tfrac{1}{2}\theta)^{-\frac{1}{2}}. \tag{6.62}$$

The relation (6.61) differs from the simple wave relation

$$v = 2\{(1+\theta)^{\frac{1}{2}}-1\}, \tag{6.63}$$

which is obtained by eliminating $\bar{F}$ from (4.14) and (4.15). This relation is the basis for the sharp-pulse approximation for any boreless pulse moving into an undisturbed region. However, as is well known, in the small amplitude limit when $\theta \ll 1$, (6.63) approximates (6.61) to within an error $O(\theta^3)$ or, equivalently, even after the passage of a weak bore, the slow Riemann function S can still be approximated by $h^{\frac{1}{2}}$ to within an error $O(\theta^3)$. More generally, for a weak bore moving into a non-uniform region, (6.63) automatically satisfies the last condition in (6.58) to within an error $O(\theta^3)$. Consequently, at all wavelets which have not coalesced into a bore, the flow is still described to a first approximation by (6.7)–(6.9). The remaining bore condition implies that, if $\alpha^+(x)$ and $\alpha^-(x)$ denote the characteristic wavelets immediately ahead and behind the bore as it passes the station x at time $t = B(x)$, then, to a first approximation,

$$h^{\frac{1}{2}}B'(x) = 1-\tfrac{3}{4}\{f(\alpha^+)+f(\alpha^-)\}H^{-\frac{5}{4}}. \tag{6.64}$$

Condition (6.64) is supplemented by the further conditions, obtained from equation (6.9), that

$$B-\int_0^x h^{-\frac{1}{2}}\,ds = \tau_0\left(\alpha^+-\frac{3}{2}\frac{f_0(\alpha^+)}{\lambda_0}\int_0^x H^{-\frac{7}{4}}\,ds\right) \tag{6.65}$$

$$= \tau_0\left(\alpha^--\frac{3}{2}\frac{f_0(\alpha^-)}{\lambda_0}\int_0^x H^{-\frac{7}{4}}\,ds\right). \tag{6.66}$$

These state that both the wavelets α^+ and α^- are at the bore $t = B(x)$ at the same instant. Once $f(\alpha)$ and $h(x)$ are known, (6.64) determines $B(x)$, $\alpha^+(x)$ and $\alpha^-(x)$.

In general, (6.64)–(6.66) must be integrated numerically. However, for a bore moving into an undisturbed region where $f(\alpha^+) \equiv 0$, these equations can be integrated without specifying the forms of the signal function $f(\alpha)$ and the depth variation $H(x)$. If $B(x)$ is eliminated from (6.64) and (6.66), an ordinary differential equation is obtained for $\alpha = \alpha^-(x)$ at the bore. This integrates to give the relation,

$$f_0(\alpha^-) = \left(\frac{4}{3}\frac{\lambda_0}{d}\int_0^{\alpha^-} f_0\,dr\right)^{\frac{1}{2}}, \tag{6.67}$$

which, once $f_0(\alpha)$ and $d(x)$ are known, determines $\alpha^-(x)$. Then, behind the bore,

$$\theta = v = f_0(\alpha^-)H^{-\frac{5}{4}}. \tag{6.68}$$

As a check on (6.67), note that it predicts that a bore forms at the front $\alpha^- = 0$ at a station $x = x_F$, given by

$$\int_0^{x_F} H^{-\frac{7}{4}}\,ds = \frac{2}{3}\frac{\lambda_0}{f_0'(0)}. \tag{6.69}$$

According to (6.57), this is the value of x at which $\Omega = 0$ at the front. The formula (6.67) for $\alpha^-(x)$ predicts that once a bore forms at the front and travels over a region where the amplitude dispersion length $d(x)$ (defined by (6.55)) remains finite, it can only be reached by those wavelets at which $f_0 > 0$. Consequently, if $\alpha = \alpha_0$ is the next zero of f_0, α^- in (6.67) varies somewhere in the range $0 \leqslant \alpha < \alpha_0$ as long as d is finite.

When the bore travels into a region of unlimited extent where H remains bounded, so that the small amplitude assumption remains valid, then, as d increases without bound, (6.67) and (6.68) predicts that at the bore the decay in

$$v = \theta \sim \left(\frac{4}{3}\int_0^{\alpha_0} f_0\,dr\right)^{\frac{1}{2}} H^{-\frac{5}{4}}\left(\frac{d}{\lambda_0}\right)^{-\frac{1}{2}}. \tag{6.70}$$

In addition, according to (6.66) and (6.67), the limiting trajectory of the bore is given by

$$t = B(x) \sim t_0(x)+\tau_0\left[\alpha_0-\left(3\int_0^{\alpha_0} f_0\,dr\right)^{\frac{1}{2}}\left(\frac{d}{\lambda_0}\right)^{\frac{1}{2}}\right]. \tag{6.71}$$

Behind the bore, the wave is fully amplitude dispersed, and

$$v = \theta \sim \frac{2}{3}\left(\alpha_0 - \overline{\frac{t-t_0}{\tau_0}}\right) H^{-\frac{5}{4}} \left(\frac{d}{\lambda_0}\right)^{-1} \quad \text{for} \quad B(x) \leqslant t \leqslant t_0 + \tau_0 \alpha_0. \tag{6.72}$$

The bore has two major effects on the pulse. It attenuates it, and, because the bore moves faster than the acoustic speed at which the wavelet α_0 travels, increases its duration at any x. In fact, according to (6.71), the duration of that part of the pulse, which before formation lasts for a time $\tau_0 \alpha_0$, increases without bound like

$$\tau_0 \left(3 \int_0^{\alpha_0} f_0 \, dr\right)^{\frac{1}{2}} \left(\frac{d}{\lambda_0}\right)^{\frac{1}{2}}, \tag{6.73}$$

as $d/\lambda_0 \to \infty$. This means that ultimately, unless $\omega(x) = O((d/\lambda_0)^{-\frac{1}{2}})$ as $d/\lambda_0 \to \infty$, the pulse is no longer short in the sense (4.2), and the effect of reflected energy must be taken into account. However, in practice, by the time the pulse approximation becomes invalid, the amplitude of the pulse may be so low that conditions in the pulse are of no interest.

As a boreless pulse approaches a shoreline, v and θ increase without bound, so that ultimately the small amplitude theory is not applicable. Since the effect of bores is to attenuate a pulse, is it possible for a pulse with bores to be attenuated rapidly enough for it to remain a small amplitude pulse right up to the shoreline? The answer is no, because, for this to be so, v and θ would have to remain bounded at the bore, and, according to (6.67) and (6.68), this is not possible, since

$$H^{-\frac{5}{4}} \left(\int_0^x H^{-\frac{7}{4}} \, ds\right)^{-\frac{1}{2}} \to \infty \quad \text{as} \quad x \to L. \tag{6.74}$$

However, even though the final stages of the pulse and bore run-up must be described by a finite amplitude theory, for beaches where $d \to \infty$ as $x \to L$ (which means that $m > \frac{2}{7}$ when H is given by (6.53)), the pulse may be completely amplitude dispersed while it can still be described by the small amplitude theory. Then, the study of the final run-up of such pulses is greatly simplified, because the profile of the pulse as it begins this last climb is of a definite form, rather than arbitrary. Below we derive conditions under which such fully dispersed small amplitude pulses can occur for polynomial beaches with $m > \frac{2}{7}$.

For definiteness, we consider a pulse of elevation, for which

$$f_0(\alpha) > 0, \quad \text{for} \quad 0 < \alpha < 1 (= \alpha_0) \quad \text{and} \quad f_0(0) = f_0(1) = 0, \tag{6.75}$$

and for which $f_0'(\alpha)$ has one zero at $\alpha = \alpha_s$. We also restrict

$$f_0''(\alpha) < 0, \tag{6.76}$$

which ensures that the only bore to form is at the front of the pulse.† Moreover, we suppose that the pulse is still described by the small-amplitude theory, which neglects the effect of frequency dispersion as it passes a station $x = x_\epsilon$, where $H = \epsilon \ll 1$. This is so if,

$$\text{at} \quad x_\epsilon, \quad \text{(i)}\ |\theta| \ll 1, \quad \text{(ii)}\ y \ll 1, \quad \text{and} \quad \text{(iii)}\ |\tau\omega| \ll 1, \tag{6.77}$$

† This pulse might be the whole wave or just the head of a more complex wave.

where the duration of the pulse $\tau(x)$ is now measured from the arrival of the bore to the arrival of the wavelet $\alpha = 1$. For $x_\epsilon \leqslant x \leqslant L$, it is convenient to write

$$H = \epsilon \tilde{H}(x) \quad \text{where} \quad 0 \leqslant \tilde{H} \leqslant 1. \tag{6.78}$$

In this range, according to (6.55) and (6.56), for $m > \frac{2}{7}$,

$$\frac{d}{\lambda_0} = \frac{4m}{7m-2} \delta^{-1} \epsilon^{-(7m-2)/4m} \tilde{H}^{-(7m-2)/4m} [1 + o(\epsilon)], \tag{6.79}$$

so that, to a first approximation, at $x = x_\epsilon$,

$$\Omega = \tau_0 \left[1 - \frac{6m}{7m-2} f_0'(\alpha) \delta^{-1} \epsilon^{-(7m-2)/4m}\right]. \tag{6.80}$$

According to (6.80), the pulse is completely amplitude dispersed at all those expansion wavelets which lie in the range $\alpha_S < \alpha \leqslant 1$, (at which $f_0'(\alpha) < 0$), and which have not yet caught up with the bore if

$$-f_0'(\alpha) \gg \frac{7m-2}{6m} \delta \epsilon^{(7m-2)/4m}. \tag{6.81}$$

This is because, since $$\theta_{,t} = v_{,t} = \Omega^{-1} f_0'(\alpha) H^{-\frac{1}{4}}, \tag{6.82}$$

at $x = x_\epsilon$, $$\theta_{,t} = v_{,t} \sim -\tau_0^{-1} \frac{7m-2}{6m} \delta \epsilon^{(m-1)/2m}, \tag{6.83}$$

when (6.81) holds. Hence, $\theta_{,t}$ and $v_{,t}$ are, to a first approximation, independent of t and the signal function $f_0(\alpha)$. The problem now reduces to showing that conditions (6.77) (i)–(iii) can be satisfied without violating the large slope, or large acceleration, condition (6.79). This amounts to showing that the amplitude of $f_{0,} = f_0(\alpha_S)$, and the parameter δ can be chosen so that all these conditions can be satisfied. This is easily done. If

$$f_0(\alpha_S) = o(\epsilon^{\frac{5}{4}}) \quad \text{as} \quad \epsilon \to 0, \tag{6.84}$$

then all these conditions are satisfied if

$$\delta \epsilon^{(m-1)/2m} \ll 1, \tag{6.85}$$

which is clearly possible.

Consequently, as $\epsilon \to 0$ a pulse, which at $x = 0$ satifies conditions (6.75), (6.76), (6.81) and (6.85), differs by a vanishingly small amount from a fully amplitude-dispersed pulse as it passes $x = x_\epsilon$. It can be shown that this pulse is preceeded by a bore whose trajectory is given by

$$\omega(\tau_0 - \overline{t - t_0}) = A \tilde{H}^{-(2+3m)/8m}, \tag{6.86}$$

where, because $f_0 = o(\epsilon)$ and because (6.85) holds,

$$A = \left(\frac{2m}{7m-2} \delta \epsilon^{-(3+2m)/4m} \int_0^1 f_0 \, dr\right)^{\frac{1}{2}} \ll 1. \tag{6.87}$$

Behind the bore, the flow is fully expanded, and

$$v \sim \theta \sim -\left(\frac{7m-2}{3m}\right) \omega(\tau_0 - \overline{t - t_0}). \tag{6.88}$$

The behaviour of this pulse as $\tilde{H} \to 0$ will be discussed in part 2.

To illustrate the behaviour of bores which form in the body of a pulse, we note that the bore conditions (6.64) and (6.65) can also be integrated when the signal function $f_0(\alpha)$ is anti-symmetric about $\alpha = \alpha_A$ for some range of $\alpha - \alpha_A$. If

$$f(\alpha_A+\phi) = -f(\alpha_A-\phi) \geqslant 0 \quad \text{for} \quad 0 \leqslant \phi \leqslant \alpha_A - \alpha_0, \tag{6.89}$$

where

$$f(\alpha_A) = f(\alpha_0) = 0, \tag{6.90}$$

then equations (6.64) and (6.65) are satisfied if

$$\alpha^+ = \alpha_A - \phi(x), \quad \text{and} \quad \alpha^- = \alpha_A + \phi(x), \tag{6.91}$$

where $\phi(x)$ is determined from the relation,

$$\frac{d(x)}{\lambda_0} = \frac{2}{3}\frac{\phi}{f(\alpha_A+\phi)}. \tag{6.92}$$

The bore trajectory is given by

$$t = B(x) \equiv t_0(x) + \tau_0 \alpha_A, \tag{6.93}$$

so that the bore moves with constant acoustic speed. Behind the bore,

$$\theta = v = f(\alpha_A+\phi)H^{-\frac{1}{4}}; \tag{6.94}$$

and, ahead of the bore,

$$\theta = v = -f(\alpha_A-\phi)H^{-\frac{1}{4}}. \tag{6.95}$$

Note that, from (6.92) as $d/\lambda_0 \to \infty$,

$$f(\alpha_A+\phi) \sim \tfrac{2}{3}(\alpha_A-\alpha_0)(d/\lambda_0)^{-1}, \tag{6.96}$$

so that, comparing (6.94) with (6.70), this bore is dissipated more rapidly than the bore moving into an undisturbed region.

The results presented in this paper were obtained in the course of research sponsored by Department of Defense Project THEMIS under Contract no. DAAD05-69-C-0053, and monitored by the Ballistics Research Laboratories, Aberdeen Proving Ground, Md.

We thank Dr P. A. Blythe for his helpful comments.

REFERENCES

Bascom, W. 1964 *Waves and Beaches: The Dynamics of the Ocean Surface.* New York: Doubleday.

Carrier, G. F. 1966 *J. Fluid Mech.* **24**, 641–659.

Lamb, H. 1945 *Hydrodynamics.* New York: Dover.

Seymour, B. R. & Varley, E. 1970 *Proc. Roy. Soc.* A **314**, 387–415.

Stoker, J. J. 1957 *Water Waves: The Mathematical Theory with Applications.* New York: Interscience.

Varley, E. & Cumberbatch, E. 1965 *J. Inst. Maths. Applics.* **1**, 101–112.

Varley, E. & Cumberbatch, E. 1966 *J. Inst. Maths. Applics.* **2**, 133–143.

Varley, E. & Cumberbatch, E. 1970 *J. Fluid Mech.* **43**, 513–537.

Whitham, G. B. 1953 *Comm. Pure Appl. Math.* **6**, 397–414.